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(Residential Autonomous College affiliated to University of Calcutta)

B.A./B.Sc. THIRD SEMESTER EXAMINATION, MARCH 2022
SECOND YEAR [BATCH 2020-23]

Date : 02/03/2022	**MATHEMATICS**	
Time : 11am-1pm	**Paper** : MTMA CC5	Full Marks : 50

Group A

Answer all the questions, maximum one can score 20.
(All symbols have their usual significance)

1. Let V be a finite dimensional inner product space. Show that the product of two self adjoint operators (defined on V) is self adjoint iff the two operators commute. [5]

2. A linear operator T on $\mathbb{R}^3$ with standard inner product, maps the vectors as follows:

$$\begin{aligned} T(0,1,1) &= (-1,0,1) \\ T(1,0,1) &= \frac{1}{3}(1,-1,4) \\ T(1,1,0) &= (0,1,1). \end{aligned}$$

Check whether T is orthogonal or not. [5]

3. Apply Gram-Schmidt process to the subset $S = \{\sin t, \cos t, 1\}$ of the inner product space $V = span(S)$ with the inner product $\langle f, g\rangle = \int_0^\pi f(t)g(t)dt$, to obtain an orthogonal basis for V. [6]

4. Define $f \in (\mathbb{R}^2)^*$ by $f(x,y) = 12x + 17y$ and $T : \mathbb{R}^2 \to \mathbb{R}^2$ by $T(x,y) = (9x + 2y, 15y)$. Compute $T^t f$ and $[T^t]_{\beta^*}$, where β is the standard ordered basis of $\mathbb{R}^2$. [1.5+2.5]

5. Let $W = span\,\{(0,i,2)\}$ in $\mathbb{C}^3$. Find orthonormal bases for W and $W^\perp$. [4]

Group B

Answer all the questions, maximum one can score 30.

6. (a) Let H be a subgroup of a group G. Prove that H is normal in G iff for all $a, b \in G$; $ab \in H$ implies $ba \in H$. [3]
   (b) Suppose G is a group of order 8. Show that the centre of G is nontrivial. [4]
   (c) Prove that the only proper subgroup of $(\mathbb{R}^*, \cdot)$ of finite index is $\mathbb{R}^+$, where $\mathbb{R}^* = \mathbb{R} - \{0\}$ and $\mathbb{R}^+ = \{x \in \mathbb{R} : x > 0\}$. [4]

7. (a) Give examples of two normal subgroups H, K of a group G such that $H \cong K$, but $G/H \ncong G/K$. [3]
   (b) Is the group $\mathbb{Z}_9$ a homomorphic image of $\mathbb{Z}_3 \times \mathbb{Z}_3$? Justify your answer. [2]

8. (a) Let R be a ring with 1 and $a \in R$. If $\exists$ a unique $b \in R$ such that $ab = 1$, prove that $ba = 1$. [2]
   (b) Prove that the units of $(\mathbb{Z}_{10}, +, \cdot)$ forms a cyclic group with respect to multiplication. [3]
   (c) Does there exist a ring epimorphism from $\mathbb{R}$ onto $\mathbb{Z}$? Justify your answer. [2]

9. (a) Show that 11 is an irreducible element in the ring $\mathbb{Z}\,[i]$. [3]

(b) Show that in $\mathbb{Z}_{12}$, $\overline{3}$ is prime but not irreducible. [3+2]

(c) Is $\overline{4}$ an associate of $\overline{6}$ in the ring $\mathbb{Z}_{10}$? Justify. [2]

(d) Show that $4\mathbb{Z}$ is a maximal ideal of $2\mathbb{Z}$. [3]

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